Mean-Square Response of an Infinite Bernoulli-Euler Beam to Nonstationary Random Excitation

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This paper presents expressions for the mean-square response of an infinite length Bernoulli-Euler beam, including linear viscous damping and a Winkler foundation, subjected to weakly nonstationary random excitations. The method used is the time domain approach, incorporating impulse response functions and the convolution integral. Results are compared to published solutions for the mean-square response of a finite simply supported beam analyzed using the modal approach. The comparison indicates that the infinite length beam response rise and decay, and time to reach maximum, differs significantly from that of the finite length beam.

Nomenclature

$E[\ ]$	= expectation operator
EI	= beam flexural rigidity
$G(x,t)$	= weakly stationary Gaussian random loading function
I_n	= modified Bessel function of the first kind of order n
J_n	= Bessel function of the first kind of order n
$K(\lambda,t)$	= infinite series function, Eq. (15)
L	= length of finite beam
R_{GG}	= autocorrelation function associated with G
R_{ww}	= autocorrelation function associated with w
S_0	= magnitude of uniform ideal white noise spectral density
a	= beam parameter, $a=\sqrt{EI/m}$
a_n	= infinite series coefficient, Eq. (16)
b_n	= infinite series coefficient, Eq. (18)
d	= viscous damping coefficient
e	= constant, $e=2.718...$
$g(x,t)$	= deterministic loading function
$h(x,t)$	= beam impulse response function
i	$=\sqrt{-1}$
k	= elastic (Winkler) foundation parameter
m	= beam mass per unit length
$q(x,t)$	= transverse loading per unit length
t	= time
u	= dummy spatial integration variable
$u(t)$	= unit step function
$w(x,t)$	= beam transverse deflection
x	= beam spatial variable, $x=0$ at center of beam
Γ	= the Gamma function
Ω_n	= damped modal frequency, Eq. (24)
β	= parameter associated with exponential decay, Eq. (27)
δ	= the Dirac delta function
ζ	= damping parameter, $\zeta=d/2m$
κ	= beam length parameter, Eq. (25)
λ	= parameter, $\lambda=\sqrt{\omega^2-\zeta^2}$
ξ	= dummy spatial integration variable
τ	= dummy time integration variable
ω	= beam parameter, $\omega=\sqrt{k/m}$

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Introduction

THE solution to a vibration problem which includes some form of random excitation involves specifying the structural response of the system in a statistical fashion. A typical approach is to determine the mean-square response of the system. Knowledge of the mean-square response of the system may then be used in one of several failure mechanisms to determine the life of the system, e.g., fatigue failure due to exceedance of a given level by the displacement response. Theoretical techniques exist for calculating the mean-square response of linear structural systems subjected to either stationary or nonstationary random excitation. Where possible, the assumption of stationary random excitation is made to simplify the analysis. In some situations, such as flapping vibration of helicopter blades,[1] the excitation is nonstationary, however, analytical simplification is possible by assuming that the excitation can be mathematically represented as the product of a deterministic function and a defined stationary random process.

Several studies have been published which examine the mean-square response of simple mechanical systems to nonstationary random excitation.[2-5] In the papers of Bucciarelli and Kuo[2] and Barnoski and Maurer,[3] the response of a single-degree-of-freedom system is studied, while Ahmadi and Satter[4] and Bogdanoff and Goldberg[5] examine a finite length Bernoulli-Euler beam using normal mode solutions for the beam response. It is the intent of this paper to present expressions for the mean-square response of an infinite length Bernoulli-Euler beam, including linear viscous damping and a Winkler foundation, subjected to weakly nonstationary random excitations. The method used is the time domain approach, incorporating impulse response functions and the convolution integral.

Basic Equation

The transverse vibration of an infinite Bernoulli-Euler beam is described by the solution of the partial differential equation

$$EI\frac{\partial^4 w}{\partial x^4}+kw+d\frac{\partial w}{\partial t}+m\frac{\partial^2 w}{\partial t^2}=q \tag{1}$$

on $-\infty<x<\infty$, $t>0$, and subject to suitable boundary conditions.

Equation (1) is to be solved for $w(x,t)$ when subjected to a loading function $q(x,t)$ which is weakly nonstationary and assumed to be expressible as the product of a deterministic function $g(x,t)$ and a weakly stationary Gaussian random loading function $G(x,t)$ so that

$$q(x,t)=g(x,t)G(x,t) \tag{2}$$

By making use of the convolution integral, the solution to Eq. (1) subjected to the loading represented by Eq. (2) can be written as

$$w(x,t)=\int_0^t\int_{-\infty}^{\infty} g(\xi,\tau)G(\xi,\tau)h(x-\xi,t-\tau)\mathrm{d}\xi\mathrm{d}\tau \tag{3}$$

where $h(x,t)$ represents the beam impulse response function. This function has been derived by the authors[6] by extending the work of Stadler and Shreeves.[7] A summary is presented in Appendix A. The expression for $h(x,t)$ representing the beam response to a unit impulse applied at the center of beam ($x=0$) at time $t=0$ is as follows [Eq. (A3)]:

$$h(x,t)=\frac{e^{-\zeta t}}{m\sqrt{4\pi a}}\int_0^t J_0(\lambda\sqrt{t^2-u^2})\frac{1}{\sqrt{u}}\cos\left(\frac{x^2}{4au}-\frac{\pi}{4}\right)\mathrm{d}u \tag{4}$$

Mean-Square Response

The response correlation for the displacement $w(x,t)$ can be written as

$$E[w(x_1,t_1)w(x_2,t_2)]=\int_0^t\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi_1,\tau_1)$$
$$\times g(\xi_2,\tau_2)h(x_1-\xi_1,t_1-\tau_1)h(x_2-\xi_2,t_2-\tau_2)$$
$$\times E[G(\xi_1,\tau_1)G(\xi_2,\tau_2)]\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_1\mathrm{d}\tau_2 \tag{5}$$

where $E[\ ]$ represents the expectation operator. It is assumed that $G(x,t)$, being a stationary random function with zero mean, has an autocorrection function R_{GG} that can be expressed as

$$E[G(\xi_1,\tau_1)G(\xi_2,\tau_2)]=R_{GG}(\xi_1-\xi_2,\tau_1-\tau_2) \tag{6}$$

By setting $x_1=x_2=x$ and $t_1=t_2=t$ in Eq. (5), and denoting the resulting response correlation or mean-square response as R_{ww}, a mean-square response function is obtained and is as follows:

$$R_{ww}(x,t)=\int_0^t\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi_1,\tau_1)g(\xi_2,\tau_2)h(x-\xi_1,t-\tau_1)$$
$$\times h(x-\xi_2,t-\tau_2)R_{GG}(\xi_1-\xi_2,\tau_1-\tau_2)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\tau_1\mathrm{d}\tau_2 \tag{7}$$

where $h(x,t)$ is given by Eq. (4). This expression represents the mean-square response at any point along the beam and at any instant of time. It is, however, intractable. By restricting the applicability, solutions can be obtained.

Two special cases are studied and results are compared with published results for a finite length beam. The first case is concerned with the response to shaped white noise multiplied by a deterministic unit step function, and the second case is a study of the response to shaped white noise modulated with an exponentially decaying deterministic function.

Response to Shaped White Noise with Unit Step

In this case the autocorrelation function R_{GG} is idealized as shaped white noise and has the form

$$R_{GG}(t_1-t_2)=2\pi S_0\delta(t_1-t_2) \tag{8}$$

where S_0 represents the uniform ideal white noise spectral density, and δ is the Dirac delta function. The deterministic function $g(x,t)$ is taken as a unit step function applied at the center of the beam ($x=0$) at time $t=0$ and can be expressed as

$$g(x,t)=\delta(x)u(t) \tag{9}$$

where $u(t)$ represents the unit step function.

On substitution of Eqs. (8) and (9) into the expression for the mean-square response, Eq. (7), and making use of the properties of the Dirac delta function, Eq. (7) reduces to

$$R_{ww}(x,t)=2\pi S_0\int_0^t h^2(x,t-\tau)\mathrm{d}\tau \tag{10}$$

which represents the mean-square response of the beam subjected to shaped white noise with unit step function applied at $x=0$ at time $t=0$.

By restricting the study to that of the response at the center of the beam ($x=0$), the expression for $h(x,t)$, Eq. (4), becomes

$$h(0,t)=\frac{e^{-\zeta t}}{m\sqrt{8\pi a}}\int_0^t J_0(\lambda\sqrt{t^2-u^2})\frac{1}{\sqrt{u}}\mathrm{d}u \tag{11}$$

and can be integrated to give explicit expressions, which are derived in Appendix A for different values of damping.

For the special case of damping $d=2\sqrt{km}$ (critical damping), then $\lambda=0$, $h(0,t)$ is given by Eq. (A16), and the mean-square response, Eq. (10), becomes

$$R_{ww}(0,t)=\frac{S_0}{4\zeta^2am^2}[1-e^{-2\zeta t}(1+2\zeta t)] \tag{12}$$

For the special case of both damping and foundation absent ($d=k=0$), then $\lambda=0$ and $\zeta=0$ and $h(0,t)$ is given by Eq. (A23), and the mean-square response, Eq. (10), becomes

$$R_{ww}(0,t)=\frac{S_0t^2}{2am^2} \tag{13}$$

For the case of general damping with elastic foundation, $h(0,t)$ can be expressed by an infinite series expansion [see Appendix, Eq. (A24)] by using the series form of the Bessel functions to obtain the expression

$$h(0,t)=\frac{\sqrt{t}e^{-\zeta t}}{m\sqrt{2\pi a}}K(\lambda,t) \tag{14}$$

Then $K(\lambda,t)$ has the form

$$K(\lambda,t)=1-\frac{(\lambda t)^2}{5}+\frac{(\lambda t)^4}{90}-\ldots+(-1)^na_n(\lambda t)^{2n}+\ldots \tag{15}$$

and a_n is given in terms of the Gamma function Γ as

$$a_n=\frac{\Gamma(5/4)}{2^{2n}n!\Gamma(n+5/4)},\qquad n=1,2,3,\ldots \tag{16}$$

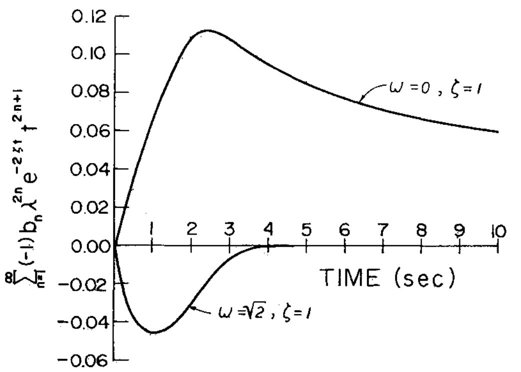

Fig. 1 Plot of the series expression in $h^2(0,t)$, Eq. (17), vs time.

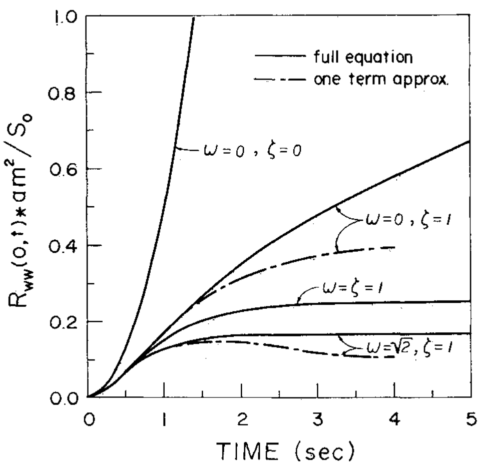

Fig. 2 Mean-square response at beam center due to shaped white noise with unit step applied at the center.

On making use of the series expression for $h(0,t)$, Eq. (14), the integrand in Eq. (10) can then be expressed in series form as

$$h^2(0,t) = \frac{te^{-2\zeta t}}{2\pi am^2}\left[1 + \sum_{n=1}^{\infty}(-1)^n b_n \lambda^{2n} t^{2n}\right] \tag{17}$$

where the coefficients b_n have been calculated in terms of a_n and are given by the following expressions:

$$b_1 = 2a_1 \approx 0.4000 \tag{18a}$$

$$b_2 = 2a_2 + a_1 a_1 \approx 0.0622 \tag{18b}$$

$$b_3 = 2a_3 + 2a_2 a_1 \approx 0.0050 \tag{18c}$$

$$b_4 = 2a_4 + 2a_1 a_3 + a_2 a_2 \approx 0.0002 \tag{18d}$$

$$b_n = 2a_n + 2a_1 a_{n-1} + 2a_2 a_{n-2} + \ldots + a_{n/2} a_{n/2}, \quad n \text{ even}$$

$$= 2a_n + 2a_2 a_{n-2} + 2a_4 a_{n-4} + \ldots + 2a_{n-1} a_1, \quad n \text{ odd} \tag{18e}$$

For λ real, $h^2(0,t)$ given by Eq. (17) is an alternating series, and for λ complex, it is a sum of positive terms, both series being convergent for all values of time. Representative curves for the two series have been numerically evaluated and are shown in Fig. 1. For λ real ($\omega = \sqrt{2}$, $\zeta = 1$) the series is negative for all times, converging to zero for large values of time. For λ complex ($\omega = 0$, $\zeta = 1$) the series peaks at time $t \approx 2.4$ s and then slowly decreases, taking a value of 0.0265 for time $t = 50$ s.

On substitution of Eq. (17) into Eq. (10), the mean-square response expression may be written as

$$R_{ww}(0,t) = \frac{S_0}{am^2}\int_0^t \tau e^{-2\zeta\tau}\left[1 + \sum_{n=1}^{\infty}(-1)^n b_n \lambda^{2n}\tau^{2n}\right]\mathrm{d}\tau \tag{19}$$

Equation (19) can then be manipulated to take the following form:

$$R_{ww}(0,t) = \frac{S_0}{4\zeta^2 am^2}\left[1 - e^{-2\zeta t}(1 + 2\zeta t)\right]$$

$$+ \frac{S_0}{am^2}\sum_{n=1}^{\infty}(-1)^n b_n \lambda^{2n}\int_0^t \tau^{2n+1} e^{-2\zeta\tau}\mathrm{d}\tau \tag{20}$$

It is noted that the infinite series term on the right-hand side of Eq. (20) contains all the effects of the elastic foundation. It is also noted that the integral in this term is a form of the incomplete Gamma function,[9] and on letting t approach infinity, the integral may be expressed in terms of the Gamma function,[8] i.e.,

$$\int_0^{\infty} \tau^{2n+1} e^{-2\zeta\tau}\mathrm{d}\tau = \Gamma(2n+2)/(2\zeta)^{2n+2}, \quad \zeta > 0$$

so that for large values of time t, Eq. (20) may be written as

$$R_{ww}(0,\infty) = \frac{S_0}{4\zeta^2 am^2}\left[1 + \sum_{n=1}^{\infty}(-1)^n b_n \Gamma(2n+2)(\lambda/2\zeta)^{2n}\right] \tag{21}$$

For small values of time, the series expression in Eq. (20) may be approximated by the first term, and then the mean-square response becomes

$$R_{ww}(0,t) = \frac{S_0}{4\zeta^2 am^2}\left[1 - e^{-2\zeta t}(1 + 2\zeta t)\right](1 - 6b_1\lambda^2/4\zeta^2)$$

$$+ \frac{S_0}{am^2}\frac{b_1\lambda^2}{(2\zeta)^2}e^{-2\zeta t}(3t^2 + 2\zeta t^3) \tag{22}$$

All of the above expressions for the mean-square response are transformed to functions of the elastic foundation parameter ω and the damping parameter ζ by multiplying by the quantity am^2/S_0. A numerical study using representative values of ζ and ω is then carried out. Curves for $am^2 R_{ww}(0,t)/S_0$ versus time have been calculated for all of the above cases and are shown in Fig. 2. For $\zeta = \omega = 0$, $\lambda = 0$, $am^2 R_{ww}(0,t)/S_0$ is unbounded. For $\zeta = 1$, $\omega = 0$, $\lambda = i$, the curve grows rapidly and appears to be bounded for large values of time. The one term approximation for this curve, Eq. (22), follows closely up to times of 1½ s and then diverges rapidly, showing an error of approximately 11% at time $t = 2$ s. For $\zeta = \omega = 1$, $\lambda = 0$, the curve approaches a steady state value of $\frac{1}{4}\zeta^2$ at approximately 3 s. When λ is real ($\zeta = 1$, $\omega = \sqrt{2}$, $\lambda = 1$), the shape of the curve closely parallels that of the preceding curve, reaching a steady state of value of approximately 0.16 and 2½ s. The one term approximation, Eq. (22), to this curve follows closely up to time $t = 1$ s and then begins to diverge, showing a lesser value, and at $t = 2$ s, there is an error of approximately 7%.

Equation (20), the series expression for the mean-square response, will now be compared with those obtained for a finite length beam.

The Finite-Length Beam

Using a modal approach, Ahmadi and Satter[4] found an approximation to the mean-square response of a simply sup-

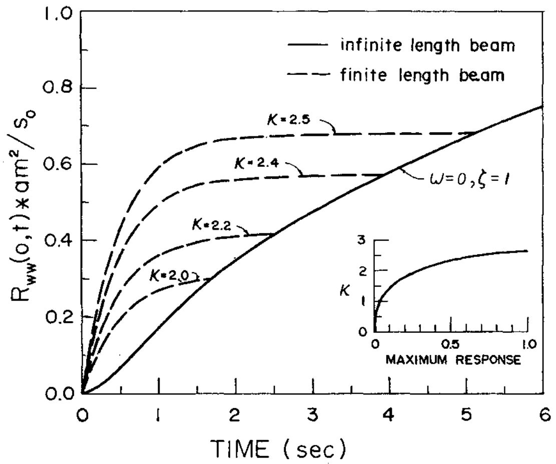

Fig. 3 Comparison of mean-square response at beam center, infinite beam and finite length beam, due to shaped white noise with unit step.

ported, finite length beam due to shaped white noise with a unit step function [their Eq. (24)] to be of the form

$$R_{ww}(0,t) = \frac{2\pi S_0}{m^2 L^2 \zeta}(1 - e^{-2\zeta t}) \sum_{n=1,3,5}^{\infty} 1/\Omega_n^2 \tag{23}$$

where L represents the length of the beam, and the damped frequency expression Ω_n is given by

$$\Omega_n^2 = a^2(n\pi/L)^4 - \zeta^2 \tag{24}$$

Equation (23) represents the response of the beam at its center, and no foundation is included. Two restrictions are imposed on the solution obtained by Ahmadi and Satter. First, light damping is assumed such that the expression for the damped frequency Ω_n remains real (in their numerical results, light damping is taken to mean $\zeta = 1$ such that $d = 2m$). Second, the mean-square response obtained is only approximate as they neglect all cross-coupled modal terms; i.e., only those terms containing products of the damped frequency $\Omega_n \Omega_p$ where $p = n$ are included in their solution.

It is pointed out by Ahmadi and Satter, on examining Eq. (24), that to keep the $1/\Omega_n^2$ terms positive and finite, the beam length must satisfy the inequality $L < \pi\sqrt{a/\zeta}$. If we let the beam length L take the form

$$L = \kappa\sqrt{a/\zeta} \tag{25}$$

where κ is a beam length parameter, and use this expression in Eq. (23), we obtain the following equation expressing the mean-square response in terms of the beam parameter κ:

$$R_{ww}(0,t) = \frac{2\pi S_0 (1 - e^{-2\zeta t})}{\zeta^2 a m^2} \sum_{n=1,3,5}^{\infty} \frac{\kappa^2}{(n\pi)^4 - \kappa^4} \tag{26}$$

This equation will now be compared to that of the infinite beam, Eq. (20), for the case of no elastic foundation ($\omega = 0$) and damping parameter $\zeta = 1$.

Comparison—White Noise with Unit Step

Two characteristics of the beam response represented by Eqs. (20) and (26) can be obtained through a comparison. The first is the mean-square response buildup with respect to time, represented by the shape of the response vs time curve. The second characteristic is the maximum response of the beam and the time to reach the maximum.

To facilitate the comparison, Eqs. (20) and (26) are multiplied by am^2/S_0, and the resulting expressions are evaluated for damping parameter $\zeta = 1$. The beam length parameter κ in Eq. (26) is varied, and Eq. (26) is evaluated for each value of κ. The resulting curves showing $am^2 R_{ww}(0,t)/S_0$ vs time are presented in Fig. 3. These curves indicate that the Ahmadi and Satter solution gives a faster rise time for the mean-square response than does the infinite beam solution. This is true for all beam lengths.

The maximum response is a function of beam length as well as time. Using the response of the inifinite beam as the upper limit for the finite beam response, a plot of κ vs maximum response is made and is included as an insert in Fig. 3. This curve indicates that the maximum response of the finite beam is extremely sensitive to beam length for $\kappa > 2$.

Response to Shaped White Noise with Exponential Decay

In this case the random loading correlation function R_{GG} is idealized as shaped white noise, as in Eq. (8). The deterministic function $g(x,t)$ is taken as an exponentially decaying function applied at the center of the beam ($x = 0$) and has the form

$$g(x,t) = \delta(x) e^{-\beta t} \tag{27}$$

where β is limited to small values so that $g(x,t)$ varies slowly.

On substitution of Eqs. (8) and (27) into the expression for the mean-square response, Eq. (7), and then making use of the properties of the Dirac delta function, the mean-square response reduces to

$$R_{ww}(x,t) = 2\pi S_0 \int_0^t e^{-2\beta\tau} h^2(x, t-\tau) d\tau \tag{28}$$

Again, by restricting the study to that of the response of the center of the beam, ($x = 0$), the expression for $h(0,t)$ is given by Eq. (11), which has explicit integrated forms derived in the Appendix and also has an infinite series form given by Eq. (14). The expression for $h^2(0,t)$ in series form is given by Eq. (17).

For the special case of damping where $d = 2\sqrt{km}$ (critical damping), then $\lambda = 0$, and on using Eq. (A16) in Eq. (28) (with $x = 0$), the mean-square response becomes

$$R_{ww}(0,t) = \frac{S_0}{4(\zeta - \beta)^2 a m^2} \{ e^{-2\beta t} - e^{-2\zeta t}[1 + 2(\zeta - \beta)t] \} \tag{29}$$

For the special case of both damping and foundation absent, $d = k = 0$, then $\omega = \zeta = 0$ and

$$R_{ww}(0,t) = \frac{S_0}{4\beta^2 a m^2} (e^{-2\beta t} + 2\beta t - 1) \tag{30}$$

For the case of general damping with elastic foundation, the expression for R_{ww} can be written, on making use of Eq. (17), in the following series form:

$$R_{ww}(0,t) = \frac{S_0 e^{-2\beta t}}{am^2} \int_0^t \tau e^{(2\beta - 2\zeta)\tau} \left[1 + \sum_{n=1}^{\infty} (-1)^n b_n \lambda^{2n} \tau^{2n} \right] d\tau \tag{31}$$

Equation (31) can be manipulated to the form:

$$R_{ww}(0,t) = \frac{S_0}{4(\zeta - \beta)^2 a m^2} \{ e^{-2\beta t} - e^{-2\zeta t}[1 + 2(\zeta - \beta)t] \} + \frac{S_0}{am^2} e^{-2\beta t} \sum_{n=1}^{\infty} (-1)^n b_n \lambda^{2n} \int_0^t \tau^{2n+1} e^{(2\beta - 2\zeta)\tau} d\tau \tag{32}$$

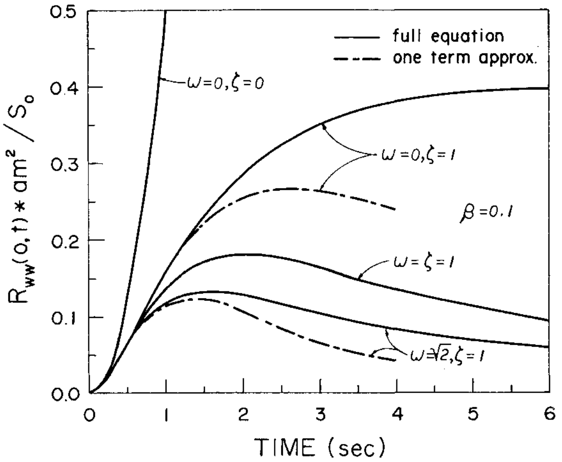

Fig. 4 Mean-square response at beam center due to shaped white noise with exponential decay applied at the center.

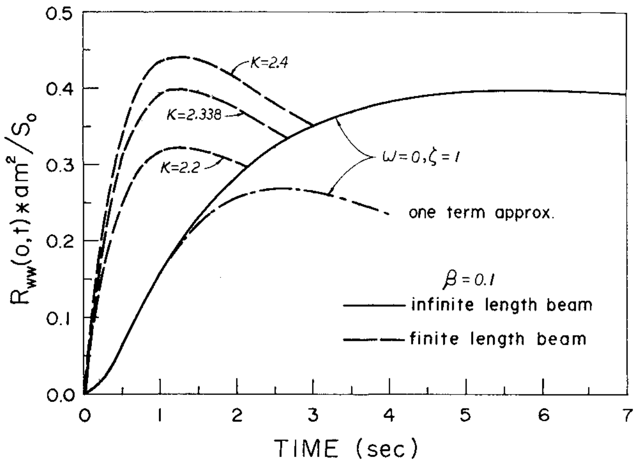

Fig. 5 Comparison of mean-square response at beam center, infinite beam and finite length beam, due to shaped white noise with exponential decay.

For time t approaching infinity, the mean-square response approaches zero for all λ.

For small values of time, the series expression in Eq. (32) may be approximated by the first term and then the mean-square response becomes

$$R_{ww}(0,t)=\frac{S_0}{4(\zeta-\beta)^2am^2}\left\{\left[e^{-2\beta t}-e^{-2\zeta t}[1+2(\zeta-\beta)t]\right]\right.$$
$$\left.\times\left(1-\frac{6b_1\lambda^2}{4(\zeta-\beta)^2}\right)+b_1\lambda^2e^{-2\zeta t}[3t^2+2(\zeta-\beta)t^3]\right\} \tag{33}$$

It is noted that this equation reduces to Eq. (22) for $\beta=0$.

A numerical study of Eqs. (29), (30), (32), and (33) is made by first multiplying the equations by am^2/S_0 and then selecting respresentative values of foundation parameter ω, damping parameter ζ, and decay parameter β. The results of this study are presented as curves in Fig. 4, where $am^2R_{ww}(0,t)/S_0$ values are plotted against time.

The curve representing Eq. (30) (no elastic foundation and no damping, $\omega=\zeta=0$, $\beta=0.1$) is unbounded as time increases. The curve representing critical damping, Eq. (29) ($\omega=\zeta=1$, $\beta=0.1$), increases to a maximum at time $t\approx2$ s and then gradually decreases as time increases. Two curves representing general damping and foundation, Eq. (32), are shown; one for λ real ($\omega=\sqrt{2}$, $\zeta=1$, $\beta=0.1$), and one for λ complex ($\omega=0$, $\zeta=1$, $\beta=0.1$). Both curves increase with time to a maximum and then gradually decrease; however, the curve for λ complex is much slower in reaching its maximum than is the curve for λ real.

Two curves respresenting the one term approximation to general damping and foundation, Eq. (33), are also shown; one for λ real ($\omega=\sqrt{2}$, $\zeta=1$, $\beta=0.1$), and one for λ complex ($\omega=0$, $\zeta=1$, $\beta=0.1$). These one term approximations agree very closely to the full expression curves for time t up to 1½ s, and then they begin to decrease more rapidly than do the full expression curves, being approximately 17% in error at time $t=2$ s.

Equation (32), the full expression equation, will now be compared to that of a finite beam.

The Finite Length Beam

Ahmadi and Satter[4] also solved for this form of nonstationary random input (white noise with exponential decay) using, as in the previously discussed example, a simply supported finite length beam. The same restrictions of light damping and neglect of cross-coupling terms in damped frequency apply. Their solution [see Ref. 4, Eq. (26)] for load application and beam response at the center of the beam, is as follows:

$$R_{ww}(0,t)=\frac{2\pi S_0}{m^2L^2(\zeta-\beta)}(e^{-2\beta t}-e^{-2\zeta t})\sum_{n=1,3,5}^{\infty}1/\Omega_n^2 \tag{34}$$

If, as was done previously, the beam length L is parameterized as $L=\kappa\sqrt{a/\zeta}$, Eq. (34) can be cast into the following form:

$$R_{ww}(0,t)=\frac{2\pi S_0}{am^2(\zeta-\beta)\zeta}(e^{-2\beta t}-e^{-2\zeta t})\sum_{n=1,3,5}^{\infty}\frac{\kappa^2}{(n\pi)^4-\kappa^4} \tag{35}$$

This equation will now be compared to that of an infinite beam [Eq. (32)] for the case of no elastic foundation ($\omega=0$), a damping parameter $\zeta=1$, and a decay parameter $\beta=0.1$.

Comparison—White Noise with Exponential Decay

Three characteristics of the mean-square response represented by Eqs. (32) and (35) can be obtained by a comparison, i.e., rate of response buildup and decay, time to reach maximum response, and beam length associated with maximum response.

To facilitate the comparison, Eqs. (32) and (35) are first multiplied by am^2/S_0 to put them in a form dependent only on foundation parameter ω, damping parameter ζ, and decay parameter β. The resulting expressions are then evaluated for foundation parameter $\omega=0$, damping parameter $\zeta=1$, and decay parameter $\beta=0.1$. The beam length parameter κ in Eq. (35) is varied and the equation is evaluated for each value of κ. The resulting curves showing $am^2R_{ww}(0,t)/S_0$ vs time are presented in Fig. 5. These curves indicate that the Ahmadi and Satter solution gives a faster rate of response buildup and decay than does the authors' solution.

A comparison of the time to reach the maximum response indicates that it is much shorter for the finite length beam than for the infinite beam. Time to reach maximum response for the finite length beam may be found analytically from Eq. (35) to be

$$t_{\max}=\frac{1}{2(\zeta-\beta)}\ln\frac{\zeta}{\beta} \tag{36}$$

which evaluates for this case to $t_{\max}=1.279$ s. The time for the infinite beam to reach its maximum mean-square response is, from Fig. 5, approximately 6 s. It is noted that the one term approximation solution, Eq. (33), shown on

Fig. 5 for comparison more closely resembles the finite beam approximation curves.

By matching the maximum response of the finite length beam to the maximum response of the infinite length beam, a beam length parameter κ for this condition may be obtained. This matching, done numerically, gives a beam length parameter $\kappa = 2.338$ for a beam length $L = 2.338\sqrt{a/1}$, where again, $a = \sqrt{EI/m}$. It is noted that the maximum response of the finite beam is extremely sensitive to beam length for beams in this length range.

Conclusions

The mean-square response of a uniform inifinite length Bernoulli-Euler beam has been determined for nonstationary random disturbances. Conclusions drawn from the analysis are:

1) Infinite beam response serves as an upper limit to finite beam response.

2) Simple beam response equations allow for the inclusion of unrestricted damping as well as an elastic foundation. The one term approximation equations [Eqs. (22) and (33)] are valid for time t equal to or less than 1 s, while for time greater than 1 s, the full equations [Eqs. (20) and (32)] should be used.

Conclusions drawn from a comparison of the authors' solution to that of Ahmadi and Satter are:

1) The response buildup for the infinite beam under loading begins slower and peaks later than the modal analysis solution done for the finite length beam for both the unit step and exponential decay loading.

2) For the exponential decay loading, a beam length of $L = 2.338\sqrt{a/\zeta}$ for $\omega = 0$, $\zeta = 1$, and $\beta = 0.1$ gives the same maximum amplitude as the infinite beam. The finite beam maximum response is extremely sensitive to beam length for beam length parameters $\kappa > 2$.

Appendix: Impulse Response Functions

Impulse response functions for the beam equation, Eq. (1), can be obtained by adapting the work of Stadler and Shreeves.[7] In their paper, Eq. (3.13) represents a general solution to the beam equation. This equation, for beam tensile force equal to zero and initial conditions $w(x,0) = 0$ and $\partial w(x,0)/\partial t = 0$, is as follows:

$$w(x,t) = \frac{1}{m}\int_0^t e^{-\zeta(t-\tau)}\int_{-\infty}^{\infty} q(\xi,\tau) \times \left\{\frac{1}{\sqrt{4\pi a}}\int_0^t J_0[\lambda\sqrt{t^2-u^2}]\frac{1}{\sqrt{u}}\cos\left(\frac{x^2}{4au}-\frac{\pi}{4}\right)du\right\}d\xi d\tau \tag{A1}$$

Impulse response functions for the beam may be obtained by setting the transverse loading $q(x,t)$ in Eq. (A1) equal to

$$q(x,t) = \delta(x)\delta(t) \tag{A2}$$

where $\delta(\;)$ represents the Dirac delta function, so that $q(x,t)$ given by Eq. (A2) represents a unit impulsive force applied at the center of the beam $(x=0)$ at time $t=0$; i.e., the impulse response function $h(x,t)$ is, on substituting Eq. (A2) in Eq. (A1) and integrating, given by

$$h(x,t) = \frac{e^{-\zeta t}}{m\sqrt{4\pi a}}\int_0^t J_0(\lambda\sqrt{t^2-u^2})\frac{1}{\sqrt{u}}\cos\left(\frac{x^2}{4au}-\frac{\pi}{4}\right)du \tag{A3}$$

On noting that the argument of J_0 in Eq. (A3) depends on λ, and that λ is given as

$$\lambda = \sqrt{k/m - (d/2m)^2} \tag{A4}$$

and can be real, zero, or complex, the integrated form of Eq. (A3) may be studied by relating λ to different values of the damping parameter d. By defining the critical damping parameter d_c as the value of d which makes $\lambda = 0$, then

$$d_c = 2\sqrt{km} \tag{A5}$$

Four distinct cases may be studied: 1) d equal to zero (zero damping); 2) d less than d_c (subcritical damping); 3) d equal to d_c (critical damping); and 4) d greater than d_c (supercritical damping). These four cases assume that an elastic foundation (k) may exist. A fifth case may be studied in which both elastic foundation and damping are equal to zero, i.e., 5) d equal to zero, k equal to zero.

On consideration of beam deflection only at the center of the beam $(x=0)$, Eq. (A3) reduces to

$$h(0,t) = \frac{1}{m\sqrt{8\pi a}}\int_0^t J_0(\lambda\sqrt{t^2-u^2})\frac{1}{\sqrt{u}}du \tag{A6}$$

Integrated forms for this impulse response function can be obtained and are as follows.

Case 1: Zero Damping

For this case $d=0$ and λ is real and equal to ω. Equation (A6) then reduces to

$$h(0,t) = \frac{1}{m\sqrt{8\pi a}}\int_0^t J_0(\omega\sqrt{t^2-u^2})\frac{1}{\sqrt{u}}du \tag{A7}$$

A change of the variable of integration to $u = t\cos\theta$, and application of Sonine's first integral[8] to the result leads to the following expression for the impulse response:

$$h(0,t) = \frac{\sqrt{t}}{m\sqrt{8\pi a}}\left[\frac{2^{-3/4}\Gamma(1/4)}{(\omega t)^{1/4}}\right]J_{1/4}[\omega t] \tag{A8}$$

This equation is the same as presented by Stadler and Shreeves[7] in their Eq. (5.4).

On making use of the series expansion for $J_{1/4}[\;]$ (see, e.g., Ref. 8, pp. 87–90), Eq. (A8) becomes

$$h(0,t) = \frac{\sqrt{t}}{m\sqrt{2\pi a}}[1 - a_1(\omega t)^2 + a_2(\omega t)^4 - \ldots + (-1)^n a_n(\omega t)^{2n} + \ldots] \tag{A9}$$

where $n = 1,2,3,\ldots$, and a_n is given by

$$a_n = \frac{\Gamma(5/4)}{2^{2n}n!\Gamma(n+5/4)} \tag{A10}$$

Case 2: Subcritical Damping

For this case $d < d_c$ and λ is real and given by the following expression:

$$\lambda = \sqrt{\omega^2 - \zeta^2} \tag{A11}$$

Equation (A6) then reduces to

$$h(0,t) = \frac{e^{-\zeta t}}{m\sqrt{8\pi a}}\int_0^t J_0(\lambda\sqrt{t^2-u^2})\frac{1}{\sqrt{u}}du \tag{A12}$$

Following the technique used in the zero damping case to evaluate the integral in the above equation, the impulse response becomes

$$h(0,t)=\frac{\sqrt{t}e^{-\zeta t}}{m\sqrt{8\pi a}}\left[\frac{2^{-3/4}\Gamma(1/2)}{(\lambda t)^{1/4}}\right]J_{1/4}[\lambda t] \qquad \text{(A13)}$$

which can be recast on using the series expansion for $J_{1/4}[\ ]$ to obtain

$$h(0,t)=\frac{\sqrt{t}e^{-\zeta t}}{m\sqrt{2\pi a}}[1-a_1(\lambda t)^2+a_2(\lambda t)^4-\ldots$$

$$+(-1)^n a_n(\lambda t)^{2n}+\ldots] \qquad \text{(A14)}$$

where the expression for a_n is given by Eq. (A10).

Case 3: Critical Damping

For this case $\lambda=0$, and on using the relation $J_0[0]=1$, Eq. (A6) reduces to

$$h(0,t)=\frac{e^{-\zeta t}}{m\sqrt{8\pi a}}\int_0^t \frac{1}{\sqrt{u}}\,du \qquad \text{(A15)}$$

which can be integrated directly to yield the following expression for the impulse response:

$$h(0,t)=\sqrt{t}e^{-\zeta t}/m\sqrt{2\pi a} \qquad \text{(A16)}$$

Case 4: Supercritical Damping

For this case, $d>d_c$ and λ are complex and given by the following expression:

$$\lambda=i\sqrt{\zeta^2-\omega^2} \qquad \text{(A17)}$$

where $i=\sqrt{-1}$. Drawing on case 2, subcritical damping, the expression for the supercritical damping impulse response can be written as

$$h(0,t)=\frac{\sqrt{t}e^{-\zeta t}}{m\sqrt{8\pi a}}\left[\frac{2^{-3/4}\Gamma(1/4)}{(it\sqrt{\zeta^2-\omega^2})^{1/4}}\right]J_{1/4}[it\sqrt{\zeta^2-\omega^2}] \qquad \text{(A18)}$$

and on using the identity (Ref. 8, p. 88),

$$i^{-n}J_n(iz)=I_n(z) \qquad \text{(A19)}$$

Eq. (A18) may be written as

$$h(0,t)=\frac{\sqrt{t}e^{-\zeta t}}{m\sqrt{8\pi a}}\left[\frac{2^{-3/4}\Gamma(1/4)}{(t\sqrt{\zeta^2-\omega^2})^{1/4}}\right]I_{1/4}[t\sqrt{\zeta^2-\omega^2}] \qquad \text{(A20)}$$

where $I_n[\ ]$ is the modified Bessel function of the first kind of order n.

On expanding $I_{1/4}[\ ]$ in series form (see, e.g., Ref. 8, p. 88), Eq. (A20) can be written as

$$h(0,t)=\frac{\sqrt{t}e^{-\zeta t}}{m\sqrt{2\pi a}}[1+a_1(t\sqrt{\zeta^2-\omega^2})^2$$

$$+a_2(t\sqrt{\zeta^2-\omega^2})^4+\ldots+a_n(t\sqrt{\zeta^2-\omega^2})^{2n}+\ldots] \qquad \text{(A21)}$$

where the expression for a_n is given by Eq. (A10). By using the complex expression for λ given by Eq. (A17), the series expression [Eq. (A21)], can be written as

$$h(0,t)=\frac{\sqrt{t}e^{-\zeta t}}{m\sqrt{2\pi a}}[1-a_1(\lambda t)^2+a_2(\lambda t)^4$$

$$-\ldots+(-1)^n a_n(\lambda t)^{2n}+\ldots] \qquad \text{(A22)}$$

Case 5: Zero Damping, No Elastic Foundation

For this case $k=0$, $d=0$, and $\lambda=0$, from Eq. (A9), the expression for the impulse response is given as

$$h(0,t)=\sqrt{t}/m\sqrt{2\pi a} \qquad \text{(A23)}$$

Summary

In their series form, the five cases can be written in compact form as

$$h(0,t)=\frac{\sqrt{t}e^{-\zeta t}}{m\sqrt{2\pi a}}K(\lambda,t) \qquad \text{(A24)}$$

where the expression for $K(\lambda,t)$ is given by

$$K(\lambda,t)=1+\sum_{n=1}^{\infty}(-1)^n a_n(\lambda t)^{2n} \qquad \text{(A25)}$$

the expression for a_n is given by Eq. (A10), and the values of λ for the five different case are summarized as follows:

Case 1: $\lambda=\omega$, zero damping;

Case 2: $\lambda=\sqrt{\omega^2-\zeta^2}$, subcritial damping;

Case 3: $\lambda=0$, critical damping;

Case 4: $\lambda=i\sqrt{\zeta^2-\omega^2}$, supercritical damping; and

Case 5: $\lambda=\omega=0$, zero damping, zero elastic foundation constant.

References

[1]Gaonkar, G. H. and Hohenemser, K. H., "Stochastic Properties of Turbulence Excited Rotor Blade Vibrations," *AIAA Journal*, Vol. 9, March 1971, pp. 419–424.

[2]Bucciarelli, L. L. and Kuo, C., "Mean-Square Response of a Second-Order System to Nonstationary, Random Excitation," *Journal of Applied Mechanics*, ASME, Series E, Vol. 37, 1970, pp. 612–616.

[3]Barnoski, R. L. and Maurer, J. R., "Mean Square Response of Simple Mechanical Systems to Non-Stationary Random Excitation," *Journal of Applied Mechanics*, ASME Series E, Vol. 36, 1969, pp. 221–227.

[4]Ahmadi, G. and Satter, M. A., "Mean-Square Response of Beams to Non-Stationary Random Excitation," *AIAA Journal*, Vol. 13, Aug. 1975, pp. 1097–1100.

[5]Bogdanoff, J. L. and Goldberg, J. E., "On the Bernoulli-Euler Beam Theory with Random Excitation," *Journal of Aerospace Sciences*, Vol. 27, May 1960, pp. 371–376.

[6]Maceyko, M. K., "Mean-Square Response of an Infinite Bernoulli-Euler Beam to Non-Stationary Random Excitation," M.S. Thesis, San Diego State Univ., CA, April 1983.

[7]Stadler, W. and Shreeves, R. W., "The Transient and Steady-State Response of the Infinite Bernoulli-Euler Beam with Damping and an Elastic Foundation," *Quaterly Journal of Mechanics and Applied Mathematics*, Vol. 23, 1970, pp. 197–208.

[8]Bowman, F., *Introduction to Bessel Functions*, Dover Publications, Inc., New York, 1958, pp. 68–69, 87–88.

[9]Davis, P. J., "Gamma Function and Related Functions," *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun, Dover Publications, Inc., New York, 1965, pp. 265–260.